1. Consider the following initial-boundary value problem for the heat equation:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0 \quad \text{for} \quad x \in (0,1),\, t > 0, \\ u(x,0) = \sin^2(2\pi x), \\ u(0,t) = u(1,t) = 0. \end{cases}$$

Find the solution u expressed as a trigonometric series.

2. Consider the following initial-boundary value problem for the Klein–Gordon equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) - u(t,x) = 0 \quad \text{for} \quad x \in (0,1),\, t > 0, \\ u(x,0) = x(1-x), \quad \frac{\partial u}{\partial t}(x,0) = 0, \\ u(0,t) = u(1,t) = 0. \end{cases}$$

Find the solution u expressed as a trigonometric series.

3. Consider the following initial-boundary value problem for the Schrödinger equation:

$$\begin{cases} i\frac{\partial u}{\partial t}(x,t) + \frac{\partial^2 u}{\partial x^2}(x,t) = 0 \quad \text{for} \quad x \in (0,1),\, t > 0, \\ u(x,0) = \sin^2(\pi x), \\ u(0,t) = u(1,t) = 0. \end{cases}$$

Find the solution u expressed as a trigonometric series.

4. Let ψ be the solution to the following initial value problem for the biharmonic heat equation on the whole line:

$$\begin{cases} \frac{\partial \psi}{\partial t}(x,t) + \frac{\partial^4 \psi}{\partial x^4}(x,t) = 0 \quad \text{for} \quad x \in \mathbb{R},\, t > 0, \\ \psi(x,0) = \psi_0(x). \end{cases}$$

Show that the solution is given by the formula

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \frac{1}{t^{\frac{1}{4}}} \int_{-\infty}^{+\infty} G\left(\frac{x-y}{t^{\frac{1}{4}}}\right) \psi_0(y)\, dy,$$

where $G : \mathbb{R} \to \mathbb{R}$ is the function for which its Fourier transform is given by

$$\hat{G}(a) = e^{-a^4}.$$

(note that, a priori, the inverse Fourier transform of the above expression should be a function $G : \mathbb{R} \to \mathbb{C}$; show that G is indeed real valued, i.e. that $\overline{G(x)} = G(x)$ for any $x \in \mathbb{R}$.)

5. Let us consider the following *semi-infinite* initial-boundary value problem for the heat equation:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x) \quad \text{for} \quad x \in (0,+\infty),\, t>0, \\ u(x,0) = 0, \\ u(0,t) = 0, \end{cases}$$

where

$$f(x) = xe^{-\frac{x^2}{2}}.$$

By extending $u(x,t)$ and $f(x)$ as *odd* functions of $x \in \mathbb{R}$, solve the above problem by applying the Fourier transform in the x-variable. Verify that the solution $u(x,t)$ that you get in this way is indeed odd and that $u(x,t)$ satisfies the required boundary condition at $x=0$ (this should be automatically true for continuous odd functions).

6 (extra). Let $V : \mathbb{R} \to \mathbb{R}$ be a smooth function (which we will call the *potential*) and let $u : \mathbb{R} \times [0,+\infty) \to \mathbb{C}$ be a solution to the Schrödinger equation:

$$i\frac{\partial u}{\partial t}(x,t) + \frac{\partial^2 u}{\partial x^2}(x,t) - V(x)u(x,t) = 0.$$

We will assume that, at any time $t \geqslant 0$, we have that $u(x,t), \frac{\partial u}{\partial x}(x,t) \to 0$ as $x \to \pm\infty$.

(a) In the special case when $V(x) = 0$, find an expression for u if the initial data at $t=0$ is given by

$$u(x,0) = e^{i\lambda x - \frac{x^2}{2}}.$$

(b) In the general case (i.e. when V is not necessarily zero), show that the quantity

$$\int_{-\infty}^{+\infty} |u(x,t)|^2 \, dx$$

is constant in time (this motivates the interpretation of $|u(x,t)|^2$ as the probability density of the particle described by u). *Hint: Use the fact that* $|u|^2 = Re\{u \cdot \bar{u}\}$ *and show first that* $\partial_t |u|^2 = 2Re\{\partial tu \cdot \bar{u}\}$. *Then, use the equation to reexpress* $\partial_t u$ *and integrate by parts in* x *if necessary. Note that* $\partial(Re(f)) = Re(\partial f)$ *and* $\partial \bar{f} = \overline{\partial f}$.

(c) Show that the total energy of u, defined by

$$E[u](t) = \int_{-\infty}^{+\infty} \left(\left| \frac{\partial u}{\partial x}(x,t) \right|^2 + V(x) \left| u(x,t) \right|^2 \right) dx$$

is also constant in time. *Hint: Differentiate the above expression in* t *like before, integrate by parts with respect to the* ∂_x*-derivatives and use the equation to substitute for the* $\frac{\partial^2 u}{\partial x^2}$ *term.*

Solutions

1. We are given the initial-boundary value problem for the heat equation:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0, & x \in (0,1),\ t > 0, \\ u(x,0) = \sin^2(2\pi x), & x \in (0,1), \\ u(0,t) = u(1,t) = 0, & t > 0. \end{cases}$$

In view of the Dirchlet boundary conditions, we look for a solution of the form:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin(n\pi x)$$

(since, as we usually do in such problems, we extend the solution as an odd, 2-periodic function of $x \in \mathbb{R}$). We substitute this form into the heat equation:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} b_n'(t)\sin(n\pi x), \quad \frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} -n^2\pi^2 b_n(t)\sin(n\pi x).$$

Hence the PDE becomes:

$$\sum_{n=1}^{\infty} b_n'(t)\sin(n\pi x) = \sum_{n=1}^{\infty} -n^2\pi^2 b_n(t)\sin(n\pi x).$$

By orthogonality of the sine functions, we get for all n:

$$b_n'(t) = -n^2\pi^2 b_n(t) \quad \Rightarrow \quad b_n(t) = b_n(0)e^{-n^2\pi^2 t},$$

where $b_n(0)$ is determined by the initial condition. We use the initial condition:

$$u(x,0) = \sin^2(2\pi x).$$

We want to expand $\sin^2(2\pi x)$ in the sine basis:

$$\sin^2(2\pi x) = \sum_{n=1}^{\infty} b_n(0)\sin(n\pi x).$$

The Fourier sine coefficients are given by:

$$b_n(0) = 2\int_0^1 \sin^2(2\pi x)\sin(n\pi x)\,dx.$$

We now simplify the integrand using the trigonometric identity:

$$\sin^2(2\pi x) = \frac{1-\cos(4\pi x)}{2}.$$

Thus,

$$b_n(0) = 2\int_0^1 \left(\frac{1-\cos(4\pi x)}{2}\right)\sin(n\pi x)\,dx = \int_0^1 (1-\cos(4\pi x))\sin(n\pi x)\,dx.$$

Now we split the integral:

$$b_n(0) = \int_0^1 \sin(n\pi x)\,dx - \int_0^1 \cos(4\pi x)\sin(n\pi x)\,dx.$$

- The first integral is:

$$\int_0^1 \sin(n\pi x)\,dx = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \frac{2}{n\pi}, & \text{if } n \text{ is odd.} \end{cases}$$

- The second integral is:

$$\begin{aligned}
\int_0^1 \cos(4\pi x)\sin(n\pi x)\,dx &= \int_0^1 \frac{e^{4\pi x i}+e^{-4\pi x i}}{2}\cdot\frac{e^{n\pi x i}-e^{-n\pi x i}}{2i}\,dx \\
&= \frac{1}{4i}\int_0^1 \left(e^{(4+n)\pi x i} - e^{(4-n)\pi x i} + e^{(-4+n)\pi x i} - e^{(-4-n)\pi x i}\right)\,dx \\
&= \begin{cases} \frac{1}{4i}\left(\frac{e^{(4+n)\pi i}-1}{(4+n)\pi i} - \frac{e^{(4-n)\pi i}-1}{(4-n)\pi i} + \frac{e^{(-4+n)\pi i}-1}{(-4+n)\pi i} - \frac{e^{(-4-n)\pi i}-1}{(-4-n)\pi i}\right), & \text{if } n \neq 4, \\ \frac{1}{4i}\left(\frac{e^{8\pi i}-1}{8\pi i} - 1 + 1 - \frac{e^{-8\pi i}-1}{-8\pi i}\right), & \text{if } n = 4, \end{cases} \\
&= \begin{cases} (1-(-1)^n)\,\frac{n}{\pi(n^2-16)}, & \text{if } n \neq 4, \\ 0, & \text{if } n = 4. \end{cases}
\end{aligned}$$

Thus,

$$b_n(0) = \begin{cases} \frac{2}{\pi}\left(\frac{1}{n} - \frac{n}{n^2-16}\right) = -\frac{32}{\pi n(n^2-16)}, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Therefore, the solution to the heat equation is:

$$u(x,t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} -\frac{32}{\pi n(n^2-16)} e^{-n^2\pi^2 t}\sin(n\pi x).$$

2. We are given the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) - u(x,t) = 0, & x \in (0,1),\ t > 0, \\ u(x,0) = x(1-x), \\ \dfrac{\partial u}{\partial t}(x,0) = 0, \\ u(0,t) = u(1,t) = 0. \end{cases}$$

In view of the Dirichlet boundary conditions, as before, we look for a solution in the form of a Fourier sine series:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin(n\pi x).$$

This satisfies the boundary conditions $u(0,t) = u(1,t) = 0$ automatically.

We compute:

$$\frac{\partial^2 u}{\partial t^2} = \sum_{n=1}^{\infty} b_n''(t)\sin(n\pi x), \quad \frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty} n^2\pi^2 b_n(t)\sin(n\pi x).$$

Substituting into the PDE:

$$\sum_{n=1}^{\infty}\left(b_n''(t) + n^2\pi^2 b_n(t) - b_n(t)\right)\sin(n\pi x) = 0.$$

By orthogonality of $\sin(n\pi x)$, each coefficient must vanish:

$$b_n''(t) + (n^2\pi^2 + 1)\,b_n(t) = 0.$$

This is a second-order linear ODE with constant coefficients:

$$b_n''(t) + \omega_n^2 b_n(t) = 0, \quad \text{where } \omega_n = \sqrt{n^2\pi^2+1}.$$

General solution:

$$b_n(t) = A_n\cos(\omega_n t) + B_n\sin(\omega_n t).$$

We will determine A_n and B_n from the initial conditions. To this end, we first need to decompose $u(x,0)$ into a sine series: We have that

$$u(x,0) = x(1-x) = \sum_{n=1}^{+\infty} c_n\sin(n\pi x),$$

where

$$c_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx.$$

We simplify:

$$c_n = 2\int_0^1 (x-x^2)\sin(n\pi x)\,dx = 2\left(\int_0^1 x\sin(n\pi x)\,dx - \int_0^1 x^2\sin(n\pi x)\,dx\right).$$

Use known integrals:

$$\int_0^1 x\sin(n\pi x)\,dx = \frac{(-1)^{n+1}}{n\pi}, \quad \int_0^1 x^2\sin(n\pi x)\,dx = \frac{(-1)^{n+1}}{n\pi} + \frac{2\left((-1)^n - 1\right)}{(n\pi)^3}.$$

Thus:

$$c_n = 2\left(\frac{(-1)^{n+1}}{n\pi} - \left(\frac{(-1)^{n+1}}{n\pi} + \frac{2\left((-1)^n - 1\right)}{(n\pi)^3}\right)\right) = \frac{4\left(1-(-1)^n\right)}{n^3\pi^3}.$$

Going back to our initial value problem, since $u(x,0) = x(1-x)$ and $\frac{\partial u}{\partial t}(x,0) = 0$, we must have, for any n:

$$b_n(0) = c_n \quad \text{and} \quad b_n'(0) = 0.$$

Since $b_n(t) = A_n \cos(\omega_n t) + B_n \sin(\omega_n t)$, substituting in the above we get

$$A_n = c_n \quad \text{and} \quad B_n = 0.$$

So:

$$b_n(t) = c_n \cos(\omega_n t).$$

Therefore, the solution is:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4\left(1-(-1)^n\right)}{n^3\pi^3} \cos\left(\sqrt{n^2\pi^2+1}\, t\right) \sin(n\pi x).$$

3. Consider the initial-boundary value problem:

$$\begin{cases} i\,\dfrac{\partial u}{\partial t}(x,t) + \dfrac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad x \in (0,1),\ t > 0, \\ u(x,0) = \sin^2(\pi x), \\ u(0,t) = u(1,t) = 0. \end{cases}$$

In view of the Dirichlet boundary conditions, we expand (as before):

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin(n\pi x).$$

Substitute into the original problem:

$$i\sum_{n=1}^{\infty} b_n'(t) \sin(n\pi x) + \sum_{n=1}^{\infty} (-n^2\pi^2) b_n(t) \sin(n\pi x) = 0.$$

This yields for each n:

$$i\,b_n'(t) - n^2\pi^2 b_n(t) = 0 \quad \Rightarrow \quad b_n'(t) = -in^2\pi^2 b_n(t).$$

Therefore,

$$b_n(t) = b_n(0) e^{-in^2\pi^2 t}.$$

We expand the initial condition in a Fourier sine series:

$$u(x,0) = \sin^2(\pi x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x).$$

Use the identity:

$$\sin^2(\pi x) = \frac{1-\cos(2\pi x)}{2},$$

so:

$$c_n = 2\int_0^1 \sin^2(\pi x)\sin(n\pi x)\,dx = 2\int_0^1 \left(\frac{1-\cos(2\pi x)}{2}\right)\sin(n\pi x)\,dx.$$

Simplify:

$$c_n = \int_0^1 (1-\cos(2\pi x))\sin(n\pi x)\,dx.$$

This is a similar computation as in the first exercise. Note:

- $$\int_0^1 \sin(n\pi x)\,dx = \begin{cases} 0, & \text{if } n \text{ even},\\ \dfrac{2}{n\pi}, & \text{if } n \text{ odd}. \end{cases}$$
- For the second term:

$$\begin{aligned}
\int_0^1 \cos(2\pi x)\sin(n\pi x)\,dx &= \int_0^1 \frac{e^{2\pi x i}+e^{-2\pi x i}}{2}\cdot\frac{e^{n\pi x i}-e^{-n\pi x i}}{2i}\,dx\\
&= \frac{1}{4i}\int_0^1 \left(e^{(2+n)\pi x i} - e^{(2-n)\pi x i} + e^{(-2+n)\pi x i} - e^{(-2-n)\pi x i}\right)\,dx\\
&= \begin{cases} \frac{1}{4i}\left(\frac{e^{(2+n)\pi i}-1}{(2+n)\pi i} - \frac{e^{(2-n)\pi i}-1}{(2-n)\pi i} + \frac{e^{(-2+n)\pi i}-1}{(-2+n)\pi i} - \frac{e^{(-2-n)\pi i}-1}{(-2-n)\pi i}\right), & \text{if } n \neq 2,\\ \frac{1}{4i}\left(\frac{e^{4\pi i}-1}{4\pi i} - 1 + 1 - \frac{e^{-4\pi i}-1}{-4\pi i}\right), & \text{if } n = 2, \end{cases}\\
&= \begin{cases} (1-(-1)^n)\,\frac{n}{\pi(n^2-4)}, & \text{if } n \neq 2,\\ 0, & \text{if } n = 2. \end{cases}
\end{aligned}$$

Thus:

$$c_n = \begin{cases} \dfrac{2}{n\pi} - \dfrac{2n}{\pi(n^2-4)} = -\dfrac{8}{\pi}\dfrac{1}{n(n^2-4)}, & \text{if } n \text{ is odd},\\ 0, & \text{if } n \text{ is even}. \end{cases}$$

Since $u(x,0) = \sin^2(\pi x)$, we must have $b_n(0) = c_n$ for all n, therefore, the final solution:

$$u(x,t) = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} -\frac{8}{\pi}\frac{1}{n(n^2-4)} e^{-in^2\pi^2 t}\sin(n\pi x).$$

4. Consider the initial value problem for the biharmonic heat equation on the real line:

$$\begin{cases} \dfrac{\partial \psi}{\partial t}(x,t) + \dfrac{\partial^4 \psi}{\partial x^4}(x,t) = 0, \quad x \in \mathbb{R},\ t > 0, \\ \psi(x,0) = \psi_0(x). \end{cases}$$

Define the Fourier transform in x as:

$$\hat{\psi}(a,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,t) e^{-iax}\, dx.$$

Taking the Fourier transform of the original problem:

$$\frac{\partial \hat{\psi}}{\partial t}(a,t) + a^4 \hat{\psi}(a,t) = 0.$$

This is a linear ODE for each a:

$$\frac{d\hat{\psi}}{dt} = -a^4 \hat{\psi} \quad \Rightarrow \quad \hat{\psi}(a,t) = \hat{\psi}(a,0) e^{-a^4 t}.$$

In view of our initial condition $\psi(x,0) = \psi_0(x)$, we must have (after taking the Fourier transform of the initial data):

$$\hat{\psi}(a,0) = \hat{\psi}_0(a).$$

Therefore

$$\hat{\psi}(a,t) = \hat{\psi}_0(a) e^{-a^4 t}.$$

We would like to invert the Fourier transform above. Since the function G in the statement satisfies

$$\mathcal{F}[G](a) = e^{-a^4},$$

using the rescaling property of the Fourier transform we can calculate

$$e^{-a^4 t} = e^{-(t^{\frac{1}{4}} a)^4} = \mathcal{F}[G(x)](t^{\frac{1}{4}} a) = \frac{1}{t^{\frac{1}{4}}} \mathcal{F}\left[G\left(\frac{x}{t^{\frac{1}{4}}} \right) \right](a).$$

Therefore, the previous expression becomes

$$\hat{\psi}(a,t) = \mathcal{F}[\psi_0](a) \cdot \mathcal{F}\left[\frac{1}{t^{\frac{1}{4}}} G\left(\frac{x}{t^{\frac{1}{4}}} \right) \right].$$

Using the property regarding the Fourier transform of convolutions, namely that $\mathcal{F}[h * g] = \sqrt{2\pi}\mathcal{F}[h] \cdot \mathcal{F}[g]$, we therefore compute that

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{t^{\frac{1}{4}}} G\left(\frac{x-y}{t^{\frac{1}{4}}} \right) \psi_0(y)\, dy.$$

Finally it remains to show that the function G is real valued, namely that $\overline{G(x)} = G(x)$. We are given:

$$\hat{G}(a) = e^{-a^4}.$$

Taking the inverse Fourier transform:

$$G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^4} e^{iax}\, da.$$

Take complex conjugate:

$$\overline{G(x)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^4} e^{-iax}\, da.$$

Making the change of variables $s = -a$, we compute:

$$\frac{1}{\sqrt{2\pi}} \int_{+\infty}^{-\infty} e^{-s^4} e^{isx}\, d(-s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-s^4} e^{isx}\, ds = G(x)$$

So:

$$\overline{G(x)} = G(x).$$

$\boxed{\text{Thus, } G(x) \in \mathbb{R} \text{ for all } x \in \mathbb{R}.}$

5. Consider the initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) = f(x), \quad x \in (0,\infty),\ t > 0, \\ u(x,0) = 0, \\ u(0,t) = 0, \end{cases} \qquad \text{where } f(x) = xe^{-x^2/2}.$$

Extend u and f on the whole real line $x \in \mathbb{R}$ as odd functions:

$$f(x) = -f(-x) \quad \text{and} \quad u(x,t) = -u(-x,t) \quad \text{for} \quad x < 0.$$

Note that $f(x)$ is still given by the formula

$$f(x) = xe^{-\frac{x^2}{2}}$$

for all $x \in \mathbb{R}$.

Then $u(x,t)$ satisfies the heat equation on $\mathbb{R}$:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) = f(x), \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = 0. \end{cases}$$

Apply the Fourier transform in x. Let:

$$\hat{u}(a,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iax} u(x,t)\, dx$$

and similarly for $\hat{f}(a)$.

Then the PDE becomes an ODE in time:

$$\frac{\partial \hat{u}}{\partial t}(a,t) + a^2 \hat{u}(a,t) = \hat{f}(a), \qquad \hat{u}(a,0) = 0.$$

There are many ways to solve the above; one would be to use the Laplace transform, another is to use the **integrating factor method**: Multiply both sides by the integrating factor $e^{a^2 t}$:

$$e^{a^2 t} \frac{d\hat{u}}{dt} + a^2 e^{a^2 t} \hat{u} = \hat{f}(a) e^{a^2 t}.$$

The left-hand side is the derivative of the product:

$$\frac{d}{dt} \left(e^{a^2 t} \hat{u}(a,t) \right) = \hat{f}(a) e^{a^2 t}.$$

Integrate both sides from 0 to t:

$$e^{a^2 t} \hat{u}(a,t) - \hat{u}(a,0) = \hat{f}(a) \int_0^t e^{a^2 s} ds.$$

Using the initial condition $u(x,0) = 0 \Rightarrow \hat{u}(a,0) = 0$, we solve for $\hat{u}(a,t)$:

$$\hat{u}(a,t) = e^{-a^2 t} \hat{f}(a) \int_0^t e^{a^2 s} ds.$$

We now simplify the integral:

$$\int_0^t e^{a^2 s} ds = \frac{1}{a^2} \left(e^{a^2 t} - 1 \right).$$

Substitute back:

$$\hat{u}(a,t) = \frac{\hat{f}(a)}{a^2} \left(1 - e^{-a^2 t} \right).$$

Since $f(x) = xe^{-x^2/2} = \frac{d}{dx}\left(e^{-\frac{x^2}{2}}\right)$ and $\mathcal{F}[e^{-\frac{x^2}{2}}](a) = e^{-\frac{a^2}{2}}$, we compute:

$$\mathscr{F}\left[xe^{-x^2/2} \right](a) = \mathscr{F}\left[\frac{d}{dx}(e^{-x^2/2}) \right](a) = iae^{-a^2/2}.$$

Therefore:

$$\hat{f}(a) = iae^{-a^2/2}.$$

Substitute into the solution:

$$\hat{u}(a,t) = i \cdot \frac{e^{-a^2/2}}{a}\left(1 - e^{-a^2 t}\right)$$

or, after multiplying both sides with ia:

$$ia\hat{u}(a,t) = -e^{-a^2/2} + e^{-a^2(\frac{1}{2}+t)}.$$

Using the facts that

- $\mathcal{F}^{-1}[ia\hat{u}(a,t)] = \frac{\partial u}{\partial x}$,
- $\mathcal{F}^{-1}[e^{-a^2/2}](x) = e^{-x^2/2}$ and
- $\mathcal{F}^{-1}[e^{-a^2(\frac{1}{2}+t)}](x) = \mathcal{F}^{-1}[e^{-\frac{(a\sqrt{1+2t})^2}{2}}](x) = \frac{1}{\sqrt{1+2t}}e^{\frac{(x/\sqrt{1+2t})^2}{2}} = \frac{1}{\sqrt{1+2t}}e^{-\frac{x^2}{2+4t}}$,

we deduce that

$$\frac{\partial u}{\partial x}(x,t) = -e^{-x^2/2} + \frac{1}{\sqrt{1+2t}}e^{-\frac{x^2}{2+4t}}.$$

Since $u(0,t) = 0$ (since u is odd in x), integrating the above in x we infer:

$$u(x,t) = -\int_0^x e^{-y^2/2}\,dy + \frac{1}{\sqrt{1+2t}}\int_0^x e^{-\frac{y^2}{2+4t}}\,dy.$$

6. Let $V : \mathbb{R} \to \mathbb{R}$ be a smooth potential and let $u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{C}$ solve the Schrödinger equation:

$$i\frac{\partial u}{\partial t}(x,t) + \frac{\partial^2 u}{\partial x^2}(x,t) - V(x)u(x,t) = 0,$$

with the decay condition:

$$u(x,t),\ \frac{\partial u}{\partial x}(x,t) \to 0 \quad \text{as } x \to \pm\infty \text{ for all } t \geqslant 0.$$

(a) We solve the free Schrödinger equation using the Fourier transform method. Let the initial condition be:

$$u(x,0) = e^{i\lambda x - \frac{x^2}{2}},$$

which is a Gaussian modulated by a plane wave of frequency λ.

We compute its Fourier transform (which is well defined since $|e^{i\lambda x - \frac{x^2}{2}}| = e^{-\frac{x^2}{2}}$ is absolutely integrable):

$$\hat{u}(a,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,0)e^{-iax}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x - \frac{x^2}{2}}e^{-iax}dx.$$

Combine exponents:

$$\hat{u}(a,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} + i(\lambda - a)x} dx.$$

This is a standard Gaussian integral of the form:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} + ibx} dx = e^{-b^2/2}, \quad \text{for any } b \in \mathbb{R}.$$

Here, $b = \lambda - a$, so:

$$\hat{u}(a,0) = e^{-(a-\lambda)^2/2}.$$

Now, we consider the evolution under the free Schrödinger equation:

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0.$$

Taking the Fourier transform of both sides in x, we get:

$$i\frac{\partial \hat{u}}{\partial t}(a,t) = a^2 \hat{u}(a,t) \Rightarrow \frac{\partial \hat{u}}{\partial t}(a,t) = -ia^2 \hat{u}(a,t),$$

which is an ordinary differential equation in time for each fixed spatial frequency a. Solving this ODE gives:

$$\hat{u}(a,t) = \hat{u}(a,0) e^{-ia^2 t}.$$

Substituting the expression for $\hat{u}(a,0)$ that we computed above:

$$\hat{u}(a,t) = e^{-(a-\lambda)^2/2} e^{-ia^2 t}.$$

The solution u is obtained by inverting the Fourier transform:

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(a,t) e^{iax} da = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a-\lambda)^2/2} e^{-ia^2 t} e^{iax} da = \frac{e^{-\frac{\lambda^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(\lambda + ix)a - (\frac{1}{2} + it)a^2} da.$$

Using the linear change of variables $a \to z(a)$,

$$z = \sqrt{1+2it}\,a - \frac{\lambda + ix}{\sqrt{1+2it}}$$

(so that $da = \frac{dz}{\sqrt{1+2it}}$), the above integral becomes

$$\begin{aligned} u(x,t) =& \frac{e^{-\frac{\lambda^2}{2}}}{\sqrt{2\pi}} \int_\gamma e^{\frac{\lambda^2}{2(1+2it)} + \frac{i\lambda x}{1+2it} - \frac{x^2}{2(1+2it)} - \frac{1}{2}z^2} \frac{dz}{\sqrt{1+2it}} \\ =& \frac{e^{-\frac{-2it\lambda^2}{2(1+2it)} + \frac{i\lambda x}{1+2it} - \frac{x^2}{2(1+2it)}}}{\sqrt{2\pi(1+2it)}} \int_\gamma e^{-\frac{1}{2}z^2}\, dz, \end{aligned}$$

where the curve γ is the straight line

$$\gamma(a) = \sqrt{1+2it}a - \frac{\lambda + ix}{\sqrt{1+2it}}, \quad a \in \mathbb{R}.$$

Using Cauchy's theorem to shift the integral from γ to $\mathbb{R}$ (using the fact that $e^{-\frac{z^2}{2}}$ decays as $\mathrm{Re}(z) \to \pm\infty$), together with the fact that $\int_{\mathbb{R}} e^{-\frac{z^2}{2}}\, dz = \sqrt{2\pi}$, we finally obtain:

$$u(x,t) = \frac{e^{-\frac{-it\lambda^2 + i\lambda x - \frac{x^2}{2}}{2(1+2it)}}}{\sqrt{1+2it}}.$$

(b) We want to show that the L^2-norm of the solution is conserved over time:

$$\frac{d}{dt}\int_{-\infty}^{\infty} |u(x,t)|^2\, dx = 0.$$

(this is the reason why $|u(x,t)|^2$ can be physically interpreted as a probability density). Since

$$|u(x,t)|^2 = u(x,t)\overline{u(x,t)},$$

We compute:

$$\frac{\partial |u(x,t)|^2}{\partial t} = \frac{\partial u}{\partial t}\bar{u} + u\frac{\partial \bar{u}}{\partial t} = 2\mathrm{Re}\left(\frac{\partial u}{\partial t}\bar{u}\right).$$

(since, $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$, so for any two complex numbers a, b: $2\mathrm{Re}(a \cdot \bar{b}) = a \cdot \bar{b} + \bar{a} \cdot b$).

Now, we use the Schrödinger equation to substitute for $\partial_t u$:

$$\frac{\partial u}{\partial t} = -i\left(-\frac{\partial^2 u}{\partial x^2} + V(x)u\right).$$

Substituting this into the previous expression for $\partial_t |u|^2$, we get:

$$\frac{\partial |u|^2}{\partial t} = 2\mathrm{Re}\left(-i\left(-\frac{\partial^2 u}{\partial x^2} + Vu\right)\bar{u}\right).$$

We now pull out the factor of $-i$, and recall that $\mathrm{Re}(-iz) = \mathrm{Im}(z)$, so:

$$\frac{\partial |u|^2}{\partial t} = 2\mathrm{Im}\left(\frac{\partial^2 u}{\partial x^2}\bar{u} - V|u|^2\right).$$

Note that $V(x)|u|^2 \in \mathbb{R}$ (since $V(x) \in \mathbb{R}$), so its imaginary part vanishes. Thus:

$$\frac{\partial \rho}{\partial t} = 2\mathrm{Im}\left(\frac{\partial^2 u}{\partial x^2}\bar{u}\right).$$

Integrating over all $x \in \mathbb{R}$:

$$\frac{d}{dt}\int_{-\infty}^{\infty} |u(x,t)|^2 dx = 2\mathrm{Im}\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}(x,t)\,\overline{u(x,t)}\,dx.$$

We now integrate by parts. Since $u(x,t)$ and $\partial_x u(x,t)$ decay to zero as $x \to \pm\infty$, the boundary terms vanish. Hence:

$$\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}\bar{u} dx = -\int_{-\infty}^{\infty} \frac{\partial u}{\partial x}\frac{\partial \bar{u}}{\partial x} dx = -\int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x}\right|^2 dx.$$

This final integral is real, so its imaginary part is zero. Therefore:

$$\frac{d}{dt}\int_{-\infty}^{\infty} |u(x,t)|^2 dx = 0.$$

$$\boxed{\int_{-\infty}^{\infty} |u(x,t)|^2 dx = \text{constant in time}}$$

This shows that the L^2-norm of the wave function is conserved, which justifies interpreting $|u(x,t)|^2$ as a probability density.

(c) We now show that the total energy of the system is conserved in time. Define the energy functional:

$$E[u](t) = \int_{-\infty}^{\infty} \left(\left|\frac{\partial u}{\partial x}\right|^2 + V(x)|u(x,t)|^2 \right) dx$$

(physically, the first term corresponds to kinetic energy and the second to potential energy). To check whether $E[u](t)$ is conserved, we differentiate with respect to time:

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial t}\left|\frac{\partial u}{\partial x}\right|^2 + V(x)\frac{\partial}{\partial t}|u|^2 \right) dx$$

(note that we used the fact that $V(x)$ is independent of t).
For convenience, let us use subscripts u_t, u_x etc for the partial derivatives $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}$. Using the fact that we computed $\frac{\partial |u|^2}{\partial t}$ before (and similarly for the derivative of $|u_x|^2$):

$$\frac{dE}{dt} = 2\mathrm{Re}\int_{-\infty}^{\infty} (u_{xt}\overline{u_x} + V(x)u_t\bar{u})\,dx.$$

Integrating by parts in x for the term $u_{xt}\bar{u}_x$ (using the fact that u_x goes to 0 as $x \to \pm\infty$), we get

$$\frac{dE}{dt} = 2\mathrm{Re}\int_{-\infty}^{\infty} (-u_t\overline{u_{xx}} + V(x)u_t\bar{u})\,dx.$$

Next, recall from the Schrödinger equation:

$$\frac{\partial u}{\partial t} = -i(-u_{xx} + Vu).$$

Substitute u_t into the expression for $\frac{dE}{dt}$:

$$\begin{aligned}\frac{dE}{dt} =& 2\mathrm{Re}\int_{-\infty}^{\infty} (i(-u_{xx} + Vu)\bar{u}_{xx} + V(x)(-i(-u_{xx} + Vu))\bar{u})\,dx \\ =& 2\mathrm{Re}\int_{-\infty}^{\infty} \left(-i|u_{xx}|^2 + iVu\bar{u}_{xx} + iVu_{xx}\bar{u} - iV|u|^2\right) dx.\end{aligned}$$

Notice that $-i|u_{xx}|^2$ and $-iV|u|^2$ are purely imaginary (so their real part is 0), so

$$\frac{dE}{dt} = 2\mathrm{Re}\int_{-\infty}^{\infty} (iVu\bar{u}_{xx} + iVu_{xx}\bar{u})\,dx.$$

Using the fact that $2\mathrm{Re}(z) = z + \bar{z}$, we have

$$2\mathrm{Re}\,(iVu\bar{u}_{xx} + iVu_{xx}\bar{u}) = (iVu\bar{u}_{xx} + iVu_{xx}\bar{u}) + (-iV\bar{u}u_{xx} - iV\bar{u}_{xx}u) = 0,$$

so $\dfrac{dE}{dt} = 0$.